Numerical convergence of model Cauchy-Characteristic Extraction and Matching

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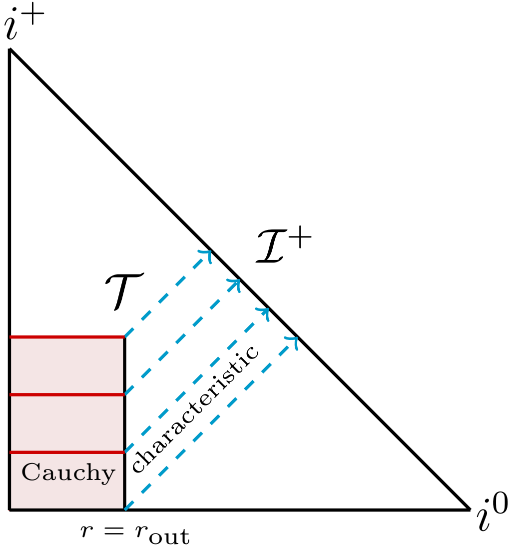
Plan

- Motivation: accurate gravitational waveform modeling
- Background: hyperbolicity and well-posedness
- A review: hyperbolicity of the characteristic system of GR
- Cauchy-Characteristic Extraction (CCE) and Matching (CCM) with toy models: energy estimates and numerical convergence
- Conclusions: Lessons for CCE and CCM in GR

Highly accurate gravitational waveform modeling



Cauchy-Characteristic Extraction and Matching

see e.g. Winicour's 2012 Living Review and references therein

Hyperbolicity

$$\boldsymbol{\mathcal{A}}^t(x^\mu)\,\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^p(x^\mu)\,\partial_p\mathbf{u} + \boldsymbol{\mathcal{S}}(\mathbf{u}, x^\mu) = 0\,, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_q)^T$, is the state vector of the system and $\boldsymbol{\mathcal{A}}^\mu$ denotes the principal part matrices, with $\det(\boldsymbol{\mathcal{A}}^t) \neq 0$. Construct the

$$\mathbf{P}^s = \left(\boldsymbol{\mathcal{A}}^t\right)^{-1} \boldsymbol{\mathcal{A}}^p\, s_p\,,$$

where s^i is an arbitrary unit spatial vector.

Degree of hyperbolicity:

- $\mathbf{P}^s$ has real eigenvalues for all s^i $\rightarrow$ (1) is weakly hyperbolic (WH)
- $\mathbf{P}^s$ is also diagonalizable for all s^i $\rightarrow$ (1) is strongly hyperbolic (SH)
- all $\boldsymbol{\mathcal{A}}^\mu$ are symmetric $\rightarrow$ (1) is symmetric hyperbolic (SYMH)

Well-posedness

The PDE problem has a unique solution that depends continuously on the given data in a suitable norm.

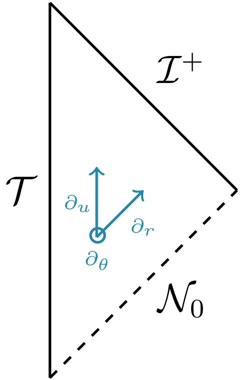
- Strongly or symmetric hyperbolic $\rightarrow$ **well-posed** IVP in the L^2 norm
- Weakly hyperbolic $\rightarrow$ **ill-posed** IVP in the L^2 norm
  possibly **weakly well-posed** in a different norm

A numerical solution **can converge** to the continuum **only** for well-posed PDE problems.

Review: hyperbolicity of the characteristic system in GR

Based on: PRD 102, 064035, TG, Hilditch, Zilhão,
PRD 105, 084055, TG, Bishop, Hilditch, Pollney, Zilhão

Bondi-like coordinates



- coordinates: u, r, θ, ϕ
- vector basis: $\partial_u\,, \partial_r\,, \partial_\theta\,, \partial_\phi$
- ∂_r is null & $\perp$ to ∂_θ and ∂_ϕ

$$g_{\mu\nu} = \begin{pmatrix} g_{uu} & g_{ur} & g_{u\theta} & g_{u\phi} \\ g_{ur} & 0 & 0 & 0 \\ g_{u\theta} & 0 & g_{\theta\theta} & g_{\theta\phi} \\ g_{u\phi} & 0 & g_{\theta\phi} & g_{\phi\phi} \end{pmatrix}$$

The vacuum Einstein Field Equations (EFE):

Characteristic evolution system: $R_{rr} = R_{r\theta} = R_{r\phi} = R_{\theta\theta} = R_{\theta\phi} = R_{\phi\phi} = 0$

Bondi, van der Burg & Sachs 1962, Winicour 2013, Cao & He 2013

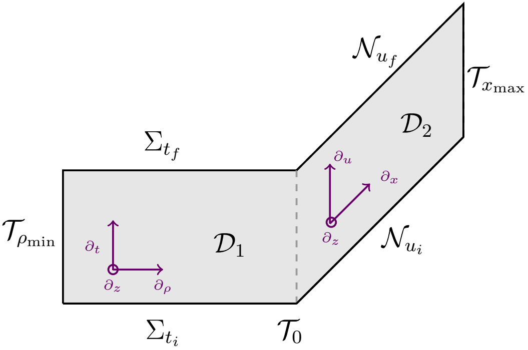
Weak hyperbolicity of the EFE in Bondi-like coordinates

- This system is WH in Bondi-Sachs and affine null coordinates: $\mathbf{P}^\theta$ and $\mathbf{P}^\phi$ are non-diagonalizable.
- The root: pure gauge structure $g^{u\theta} = g^{u\phi} = 0$
- GR[1] in all Bondi-like gauges $\rightarrow$ weakly hyperbolic PDE system.
- The CIBVP is ill-posed in the L^2 norm.
  Could it be weakly well-posed in another norm? (open question)
- How does this affect accuracy of CCE and CCM?

[1] With up to second order metric derivatives

CCE and CCM with toy models

The toy models



$$\partial_t \phi_1 = -\partial_\rho \phi_1 + \boxed{\partial_z \psi_{v1}} + \psi_1$$
$$\partial_t \psi_{v1} = -\partial_\rho \psi_{v1} + \partial_z \phi_1$$
$$\partial_t \psi_1 = \partial_\rho \psi_1 + \partial_z \psi_1$$

$$\partial_x \phi_2 = \boxed{\partial_z \psi_{v2}}$$
$$\partial_x \psi_{v2} = \partial_z \phi_2$$
$$\partial_u \psi_2 = \frac{1}{2}\partial_x \psi_2 + \partial_z \psi_2 + \psi_{v2}$$

SYMH when $\partial_z \psi_v$ is included, WH otherwise

Energy estimates

Well-posedness: there exists a unique solution **u** that
depends continuously on the given data f in an appropriate norm $||\cdot||$:

$||\mathbf{u}|| \leq Ke^{\alpha t}||f||$, for real constants $K > 1$, α, and t.

SYMH IBVP: $||\mathbf{u}_1||_{L^2} \equiv \int_{\Sigma_{t_f}} \left(\phi_1^2 + \psi_{v1}^2 + \psi_1^2\right) + \int_{\mathcal{T}_0} \left(\phi_1^2 + \psi_{v1}^2\right) + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2$

WH IBVP: $||\mathbf{u}_1||_q \equiv$
$\int_{\Sigma_{t_f}} \left[\phi_1^2 + \psi_{v1}^2 + \psi_1^2 + (\partial_z \phi_1)^2\right] + \int_{\mathcal{T}_0} \left[\phi_1^2 + \psi_{v1}^2 + (\partial_z \phi_1)^2\right] + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2$

SYMH CIBVP: $||\mathbf{u}_2||_{L^2} \equiv \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_0} \psi_2^2 + \max_{x'} \int_{\mathcal{T}_{x'}} \left(\phi_2^2 + \psi_{v2}^2\right)$

WH CIBVP: $||\mathbf{u}_2||_q \equiv \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_0} \psi_2^2 + \max_{x'} \int_{\mathcal{T}_{x'}} \left[\phi_2^2 + \psi_{v2}^2 + (\partial_z \phi_2)^2\right]$

Energy estimates

For CCE well-posedness is examined individually for the IBVP and CIBVP.

For CCM, the composite IBVP-CIBVP problem has to be examined as a whole.

<u>SYMH-SYMH:</u>
$||\mathbf{u}||_{L^2} \equiv \int_{\Sigma_{t_f}} (\phi_1^2 + \psi_{v1}^2 + \psi_1^2) + \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2 + \max_{x'} \int_{\mathcal{T}_{x'}} (\phi_2^2 + \psi_{v2}^2)$

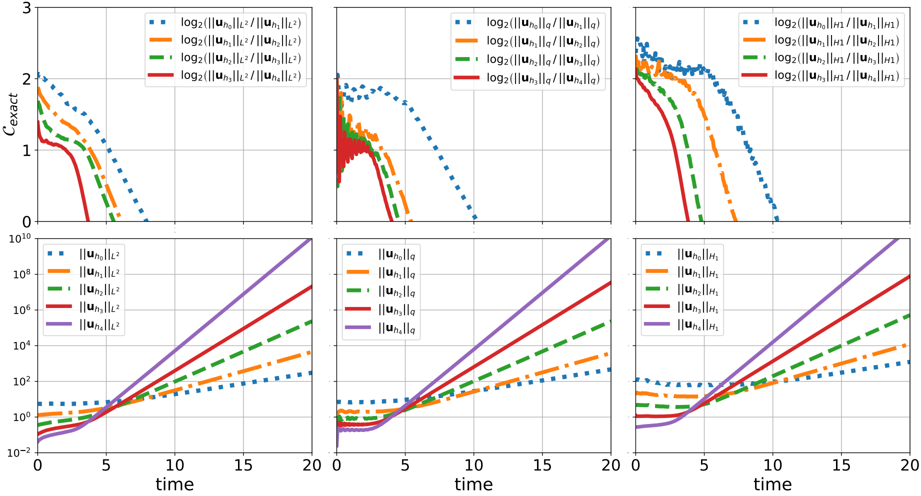
<u>WH-WH:</u> $||\mathbf{u}||_q \equiv \int_{\Sigma_{t_f}} \left[\phi_1^2 + \psi_{v1}^2 + \psi_1^2 + (\partial_z \phi_1)^2\right] + \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2 + \max_{x'} \int_{\mathcal{T}_{x'}} \left[\phi_2^2 + \psi_{v2}^2 + (\partial_z \phi_2)^2\right]$

We cannot get an energy estimate for SYMH-WH CCM due a $\int_{\mathcal{T}_0}$ term that is not controlled by given data.

Convergence tests

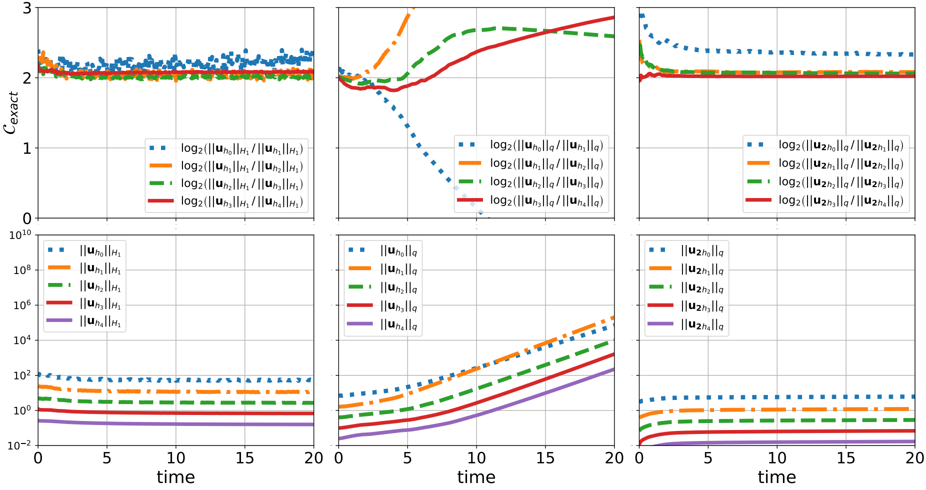
- Accuracy of numerical solution: $f - f_h = O(h^n)$
- Convergence factor: $Q = h_0^n / h_1^n = f_0 / f_1$
- High frequency given data: random noise of amplitude A_h
- We assume the exact solution $\mathbf{u} = \mathbf{0}$ and monitor $\mathcal{C}_{\mathrm{exact}} = \log_2 \frac{||\mathbf{u}_{h_0}||_{h_0}}{||\mathbf{u}_{h_1}||_{h_1}}$
- Every time we double resolution we drop A_h by $1/4$ for no derivative variables and by $1/8$ for those with derivatives $\rightarrow \mathcal{C}_{\mathrm{exact}} = 2$

Convergence tests



CCM between the SYMH IBVP and the WH CIBVP in different norms

Convergence tests



CCM between the SYMH-SYMH (left), WH-WH (middle) and the WH CIBVP (right) for CCE between SYMH-WH

Conclusions

Lessons for GR based on our CCE and CCM analysis for toy models:

- if the WH CIBVP is weakly well-posed, CCE can also be well-posed
- Is there an appropriate norm for the WH Bondi-like CIBVP?
- CCM as currently performed (SYMH-WH) is ill-posed and cannot provide convergent solutions
- Problem with error estimates for accurate waveforms with CCM
- A strongly or symmetric hyperbolic characteristic formulation is needed (with up to 2nd order metric derivatives)

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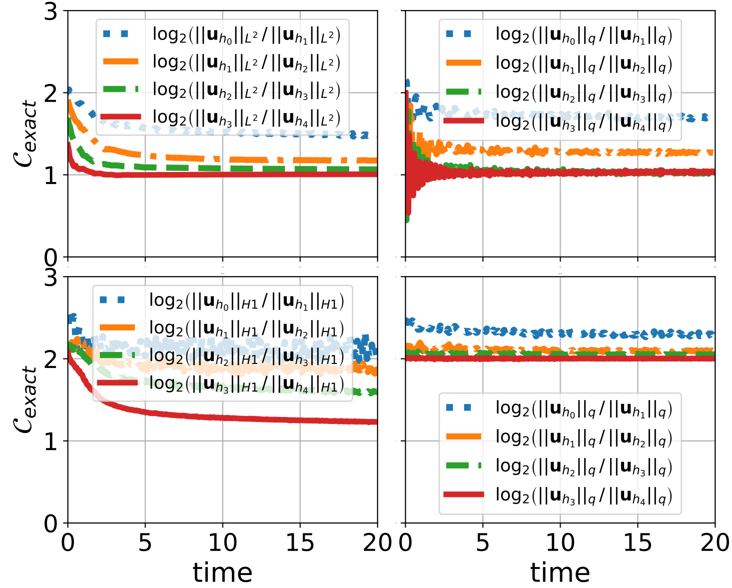
Thank you!

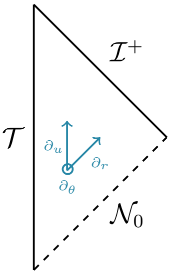
Extras



CCM between the homogeneous SYMH IBVP and the WH CIBVP in different norms

Hyperbolicity of GR in the Bondi-Sachs gauge

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du\, dr$$
$$+ 2Ur^2 e^{2\gamma} du\, d\theta - r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2\theta\, d\phi^2\right) .$$



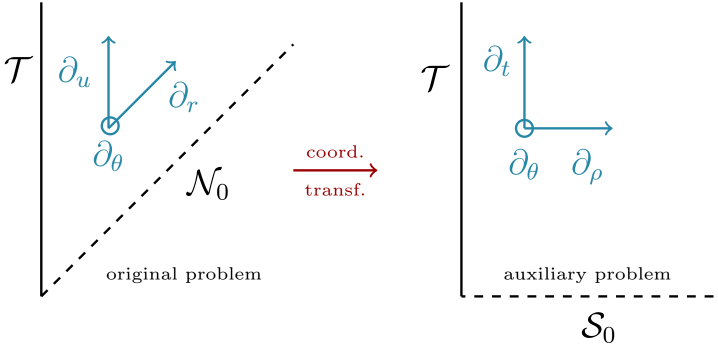
The PDE system:

$$\partial_r \beta = F_1(\partial_r \gamma) ,$$
$$\partial_r^2 U = F_2(\gamma, \beta, \partial_i\gamma, \partial_i\beta, \partial_{ij}^2\gamma, \partial_{ij}^2\beta) ,$$
$$\partial_r V = F_3(\gamma, \beta, \partial_i\gamma, \partial_i\beta, \partial_i U, \partial_{ij}^2\gamma, \partial_{ij}^2\beta, \partial_{ij}^2 U) ,$$
$$\partial_{ur}^2\gamma = F_4(\gamma, \beta, U, V, \partial_i\gamma, \partial_i\beta, \partial_i U, \partial_i V, \partial_{ij}^2\gamma, \partial_{ij}^2\beta, \partial_{ij}^2 U)$$

Linearize and first order reduction $\mathbf{u} = (\beta\, , \gamma\, , U\, , V\, , \gamma_r\, , U_r\, , \beta_\theta\, , \gamma_\theta)^T$:

$$\boldsymbol{\mathcal{A}}^u \partial_u \mathbf{u} + \boldsymbol{\mathcal{A}}^r \partial_r \mathbf{u} + \boldsymbol{\mathcal{A}}^\theta \partial_\theta \mathbf{u} + \boldsymbol{\mathcal{S}} = 0.$$

$$\det(\boldsymbol{\mathcal{A}}^u) = 0\,, \qquad \begin{aligned} u &= t - \rho\,, \\ r &= \rho\,, \end{aligned} \qquad \det(\boldsymbol{\mathcal{A}}^t) \neq 0\,.$$



$\boldsymbol{\mathcal{A}}^t\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^\rho\partial_\rho\mathbf{u} + \boldsymbol{\mathcal{A}}^\theta\partial_\theta\mathbf{u} + \boldsymbol{\mathcal{S}} = 0$, where $\boldsymbol{\mathcal{A}}^t = \boldsymbol{\mathcal{A}}^u + \boldsymbol{\mathcal{A}}^r$ and $\boldsymbol{\mathcal{A}}^\rho = \boldsymbol{\mathcal{A}}^r$.

$\mathbf{P}^\theta = \frac{1}{\rho}\left(\boldsymbol{\mathcal{A}}^t\right)^{-1}\boldsymbol{\mathcal{A}}^\theta$ is not diagonalizable.

The Bondi-Sachs system is weakly hyperbolic.

Rendall 1990, Frittelli 2005 & 2006, TG, Hilditch & Zilhão 2020

Frame independence

Focus on the angular direction:

$$\partial_t \mathbf{u} + \mathbf{B}^{\hat{\theta}} \partial_{\hat{\theta}} \mathbf{u} \simeq 0 \quad \longrightarrow \quad \partial_t \mathbf{v} + \mathbf{J}^{\hat{\theta}} \partial_{\hat{\theta}} \mathbf{v} \simeq 0\,,$$

where $\mathbf{J}^{\hat{\theta}} \equiv \mathbf{T}_{\hat{\theta}}^{-1}\, \mathbf{B}^{\hat{\theta}}\, \mathbf{T}_{\hat{\theta}}$ is the Jordan normal form and $\mathbf{v} \equiv \mathbf{T}_{\hat{\theta}}^{-1}\, \mathbf{u}$ the generalized characteristic variables. The non-trivial Jordan block yields

$$-\,\partial_t \left(2\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta \right) \simeq 0\,,$$

$$\partial_t V - \partial_\theta \left(2\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta \right) \simeq 0\,.$$

The generalized eigenvalue problem:

$$\mathbf{l}_\lambda \left(\mathbf{P}^s - \mathbf{1}\lambda \right)^M = 0\,,$$

where $M = 1, 2, \cdots$.

Gauge structure of GR

The ADM equations linearized about Minkowski:

$$\partial_t \delta\gamma_{ij} = -2\delta K_{ij} + \partial_{(i}\delta\beta_{j)}\,,$$
$$\partial_t \delta K_{ij} = -\partial_i\partial_j\delta\alpha - \tfrac{1}{2}\partial^k\partial_k\delta\gamma_{ij} - \tfrac{1}{2}\partial_i\partial_j\delta\gamma + \partial^k\partial_{(i}\delta\gamma_{j)k}\,.$$

First order reduction $\mathbf{u} = (\delta\gamma_{ij}, \delta\alpha, \delta\beta_i, \delta K_{ij}, \partial_p\delta\gamma_{ij}, \partial_p\delta\alpha, \partial_p\delta\beta_i)^T$:

$$\partial_t \mathbf{u} \simeq \mathbf{P}^s \partial_s \mathbf{u}\,, \quad \text{with} \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & 0 \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}\,.$$

Hilditch & Richter 2016

Pure gauge subsystem

Assume an arbitrary solution $g_{\mu\nu}$ of $R_{\mu\nu} = 0$.

- Infinitesimal coordinate transformation: $x^\mu \to x^\mu + \xi^\mu$
- Perturbation to the solution: $\delta g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}$
- $3+1$ split: $\Theta \equiv n_\mu \xi^\mu$, $\psi^i \equiv -\gamma^i{}_\mu \xi^\mu$

Pure gauge subsystem for flat background:

$$\partial_t \Theta = \delta\alpha \,,$$
$$\partial_t \psi_i = \delta\beta_i + \partial_i \Theta \,.$$

Given α, β_i, the pure gauge subsystem is closed.

Khokhlov & Novikov 2001

Pure gauge subsystem inheritance

Linearized ADM system:

$$\partial_t \mathbf{u} \simeq \mathbf{P}^s \partial_s \mathbf{u}\,, \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & 0 \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}\,.$$

Assume an algebraic choice for α, β_i. Pure gauge subsystem:

$$\partial_t \mathbf{v}_{gauge} \simeq \mathbf{P}^s_{gauge} \partial_s \mathbf{v}_{gauge}\,, \quad v_{gauge} = (\Theta, \psi_i)^T\,.$$

The inheritance: $\mathbf{P}_G = \mathbf{P}^s_{gauge}$

The result holds also for generic backgrounds & differential gauge choices.

Hilditch & Richter 2016

Algebraic determination of well-posedness

For the initial value problem (IVP) with constant coefficients:

$$\partial_t \mathbf{u} = \mathbf{B}^p \partial_p \mathbf{u} + \mathbf{S} \equiv \mathbf{B}^p \partial_p \mathbf{u} + \mathbf{B}\mathbf{u}\,,$$

after Fourier transforming in space ($\partial_p \to i\omega_p$):

$$\mathbf{P}(i\omega) = i\omega_p \mathbf{B}^p + \mathbf{B} \quad \longrightarrow \quad \mathbf{u}(\cdot\,, t) = e^{\mathbf{P}(i\omega)t} \hat{f}(\omega)\,.$$

If $|e^{\mathbf{P}(i\omega)t}| \leq K e^{\alpha t}$, $K \geq 1$ & $\alpha \in \mathbb{R}$ for $t \geq 0$, the IVP is well posed in L^2.

$$||\mathbf{u}(\cdot\,, t)||_{L^2} = ||e^{\mathbf{P}(i\omega)t} \hat{f}(\omega)||_{L^2} \leq K e^{\alpha t} ||\hat{f}||_{L^2} = K e^{\alpha t} ||f||_{L^2}\,.$$

If $|e^{\mathbf{P}(i\omega)t}| \leq K_1 e^{\alpha t} (1 + |\omega|^q) \longrightarrow$ well-posed in a lopsided norm (weakly).

Eigenvalues of $\mathbf{P}(i\omega)$

Gustafsson, Kreiss & Oliger "*Time Dependent Problems and Difference Methods*"
Kreiss & Lorenz "*Initial-Boundary Value Problems and the Navier-Stokes Equations*"