Hyperbolicity of GR in Bondi-like coordinates

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PhysRevD.102.064035 T.G, D. Hilditch & M. Zilhão
2111.14794 TG, N. Bishop, D. Hilditch, D. Pollney & M. Zilhão

Plan

- Background & motivation:
  numerical relativity,
  partial differential equations,
  Bondi-like coordinates

- Main result:
  hyperbolicity of GR in Bondi-like coordinates

- Well-posedness of the characteristic problem of GR

Bonus:

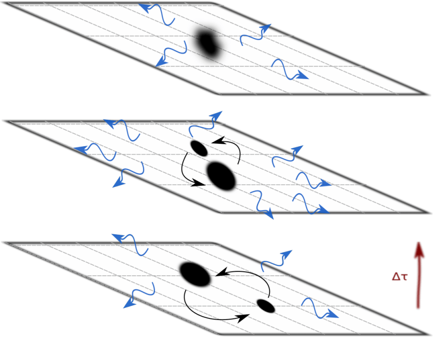
- Numerics & convergence

Background & motivation

Numerical relativity

Einstein's field equations (EFEs): $R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = \frac{8\pi G}{c^4} T_{ab}$

- Numerical (approximate) solutions to EFEs
- Finite time & space
- Time evolution (hyperbolic PDE system)
- Choose a gauge (coordinates)



Spacetime foliation

Hyperbolicity

$$\mathcal{A}^t(x^\mu)\,\partial_t \mathbf{u} + \mathcal{A}^p(x^\mu)\,\partial_p \mathbf{u} + \mathcal{S}(\mathbf{u}, x^\mu) = 0\,,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_q)^T$, is the state vector of the system and

$$\mathcal{A}^\mu = \begin{pmatrix} a^\mu_{11} & \ldots & a^\mu_{1q} \\ \vdots & \ddots & \vdots \\ a^\mu_{q1} & \ldots & a^\mu_{qq} \end{pmatrix}$$

denotes the principal part matrices, with $\det(\mathcal{A}^t) \neq 0$. Construct the

$$\mathbf{P}^s = \left(\mathcal{A}^t\right)^{-1} \mathcal{A}^p\, s_p\,,$$

where s^i is an arbitrary unit spatial vector.

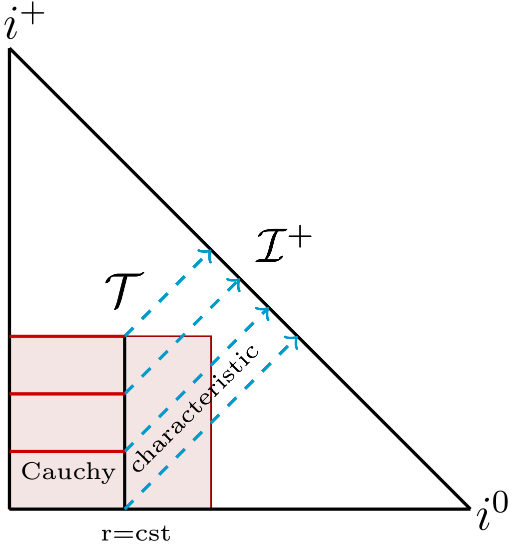
Well-posedness

The PDE problem has a unique solution that depends continuously on the given data in a suitable norm.

- Strongly hyperbolic (SH) $\rightarrow$ **well-posed** IVP in the L^2 norm
- Weakly hyperbolic (WH) $\rightarrow$ **ill-posed** IVP in the L^2 norm

A numerical solution **can converge** to the continuum **only** for well-posed PDE problems.

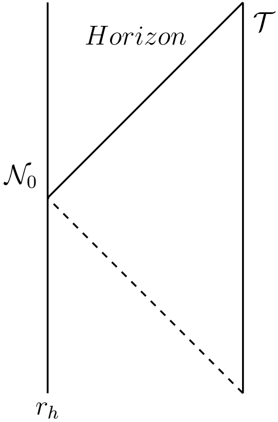
Highly accurate gravitational waveform modeling



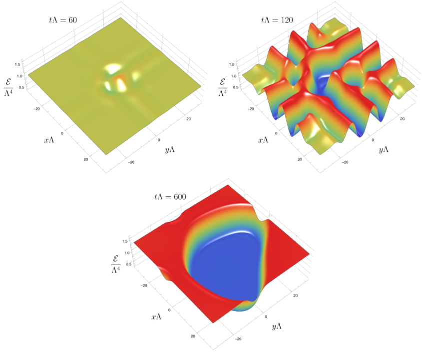
Cauchy-Characteristic extraction

see e.g. Winicour's 2012 Living Review and references therein

Holography & strongly coupled matter

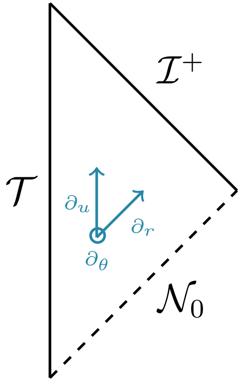


Asymptotically AdS spacetime



Model phase transitions (arXiv:2112.15478).
www.youtube.com/watch?v=qIhbpchr3gE

Bondi-like coordinates



- coordinates: u, r, θ, ϕ
- vector basis: $\partial_u\,, \partial_r\,, \partial_\theta\,, \partial_\phi$
- ∂_r is null & $\perp$ to ∂_θ and ∂_ϕ

$$g_{\mu\nu} = \begin{pmatrix} g_{uu} & g_{ur} & g_{u\theta} & g_{u\phi} \\ g_{ur} & 0 & 0 & 0 \\ g_{u\theta} & 0 & g_{\theta\theta} & g_{\theta\phi} \\ g_{u\phi} & 0 & g_{\theta\phi} & g_{\phi\phi} \end{pmatrix}$$

Vacuum Einstein's equations:

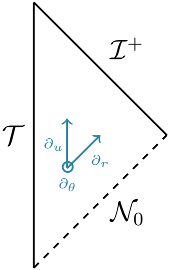
Evolution system: $R_{rr} = R_{r\theta} = R_{r\phi} = R_{\theta\theta} = R_{\theta\phi} = R_{\phi\phi} = 0$

Bondi, van der Burg & Sachs 1962, Winicour 2013, Cao & He 2013

Main part: hyperbolicity

Hyperbolicity of GR in the Bondi-Sachs gauge

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du\, dr$$
$$+ 2Ur^2 e^{2\gamma} du\, d\theta - r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2\theta\, d\phi^2\right) .$$



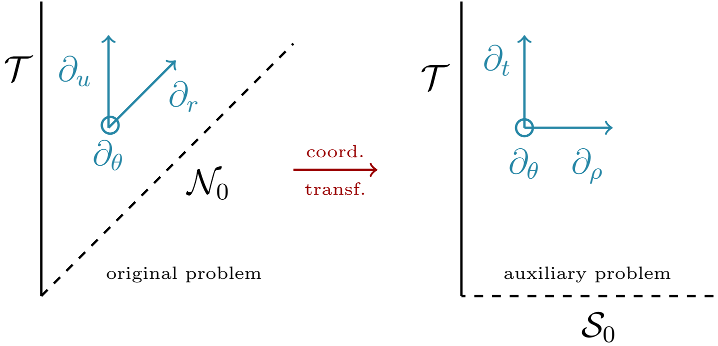
The PDE system:

$$\partial_r \beta = F_1(\partial_r \gamma)\,,$$
$$\partial_r^2 U = F_2(\gamma, \beta, \partial_i \gamma, \partial_i \beta, \partial_{ij}^2 \gamma, \partial_{ij}^2 \beta)\,,$$
$$\partial_r V = F_3(\gamma, \beta, \partial_i \gamma, \partial_i \beta, \partial_i U, \partial_{ij}^2 \gamma, \partial_{ij}^2 \beta, \partial_{ij}^2 U)\,,$$
$$\partial_{ur}^2 \gamma = F_4(\gamma, \beta, U, V, \partial_i \gamma, \partial_i \beta, \partial_i U, \partial_i V, \partial_{ij}^2 \gamma, \partial_{ij}^2 \beta, \partial_{ij}^2 U)$$

Linearize and first order reduction $\mathbf{u} = (\beta\,, \gamma\,, U\,, V\,, \gamma_r\,, U_r\,, \beta_\theta\,, \gamma_\theta)^T$:

$$\boldsymbol{\mathcal{A}}^u \partial_u \mathbf{u} + \boldsymbol{\mathcal{A}}^r \partial_r \mathbf{u} + \boldsymbol{\mathcal{A}}^\theta \partial_\theta \mathbf{u} + \boldsymbol{\mathcal{S}} = 0.$$

$$\det(\boldsymbol{\mathcal{A}}^u) = 0\,, \qquad \begin{aligned} u &= t - \rho\,, \\ r &= \rho\,, \end{aligned} \qquad \det(\boldsymbol{\mathcal{A}}^t) \neq 0\,.$$



$\boldsymbol{\mathcal{A}}^t\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^\rho\partial_\rho\mathbf{u} + \boldsymbol{\mathcal{A}}^\theta\partial_\theta\mathbf{u} + \boldsymbol{\mathcal{S}} = 0$, where $\boldsymbol{\mathcal{A}}^t = \boldsymbol{\mathcal{A}}^u + \boldsymbol{\mathcal{A}}^r$ and $\boldsymbol{\mathcal{A}}^\rho = \boldsymbol{\mathcal{A}}^r$.

$\mathbf{P}^\theta = \frac{1}{\rho}\left(\boldsymbol{\mathcal{A}}^t\right)^{-1}\boldsymbol{\mathcal{A}}^\theta$ is not diagonalizable.

The Bondi-Sachs system is weakly hyperbolic.

Rendall 1990, Frittelli 2005 & 2006, TG, Hilditch & Zilhão 2020

Gauge structure of GR

The ADM equations linearized about Minkowski:

$$
\begin{aligned}
\partial_t \delta\gamma_{ij} &= -2\delta K_{ij} + \partial_{(i}\delta\beta_{j)}\,,\\
\partial_t \delta K_{ij} &= -\partial_i\partial_j\delta\alpha - \tfrac{1}{2}\partial^k\partial_k\delta\gamma_{ij} - \tfrac{1}{2}\partial_i\partial_j\delta\gamma + \partial^k\partial_{(i}\delta\gamma_{j)k}\,.
\end{aligned}
$$

First order reduction $\mathbf{u} = (\delta\gamma_{ij}, \delta\alpha, \delta\beta_i, \delta K_{ij}, \partial_p\delta\gamma_{ij}, \partial_p\delta\alpha, \partial_p\delta\beta_i)^T$:

$$
\partial_t \mathbf{u} \simeq \mathbf{P}^s\partial_s\mathbf{u}\,, \quad \text{with} \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & 0 \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}.
$$

Hilditch & Richter 2016

Pure gauge subsystem

Assume an arbitrary solution $g_{\mu\nu}$ of $R_{\mu\nu} = 0$.

- Infinitesimal coordinate transformation: $x^\mu \to x^\mu + \xi^\mu$
- Perturbation to the solution: $\delta g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}$
- $3+1$ split: $\Theta \equiv n_\mu \xi^\mu$, $\psi^i \equiv -\gamma^i{}_\mu \xi^\mu$

Pure gauge subsystem for flat background:

$$\begin{aligned}
\partial_t \Theta &= \delta\alpha \,, \\
\partial_t \psi_i &= \delta\beta_i + \partial_i \Theta \,.
\end{aligned}$$

Given α, β_i, the pure gauge subsystem is closed.

Khokhlov & Novikov 2001

Pure gauge subsystem inheritance

Linearized ADM system:

$$\partial_t \mathbf{u} \simeq \mathbf{P}^s \partial_s \mathbf{u}\,, \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & 0 \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}.$$

Assume an algebraic choice for α, β_i. Pure gauge subsystem:

$$\partial_t \mathbf{v}_{gauge} \simeq \mathbf{P}^s_{gauge} \partial_s \mathbf{v}_{gauge}\,, \quad v_{gauge} = (\Theta, \psi_i)^T\,.$$

The inheritance: $\mathbf{P}_G = \mathbf{P}^s_{gauge}$

The result holds also for generic backgrounds & differential gauge choices.

Hilditch & Richter 2016

Gauge structure of Bondi-like coordinates

Non-diagonalizable $\mathbf{P}_G$ along the θ, ϕ directions.

$$g^{u\theta} = g^{u\phi} = 0$$

Mapping between Bondi-like and ADM equations.

GR formulation with up to 2nd order metric derivatives:

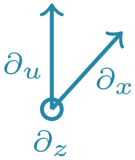
- In any Bondi-like gauge the PDE system is only WH.
- This CIBVP is ill-posed in the L^2 norm.
- CCE accuracy?

TG, Bishop, Hilditch, Pollney & Zilhão 2021

Main part: well-posedness

The toy models

$$\partial_x \phi = -S_\phi$$
$$\partial_x \psi_v - \boxed{\partial_z \phi} = -S_{\psi_v}$$
$$\partial_u \psi - F(x)\partial_x \psi - \partial_z \psi = -S_\psi$$

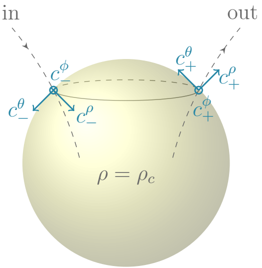


- SH well-posed in $||\mathbf{u}||^2_{L^2(\mathcal{D})} = \int_{\mathcal{D}} \left(\phi^2 + \psi_v^2 + \psi^2\right)$
- WH well-posed in $||\mathbf{u}||^2_{q(\mathcal{D})} = \int_{\mathcal{D}} \left(\phi^2 + \psi_v^2 + \psi^2 + (\partial_z \phi)^2\right)$

  For $S_\phi = \psi_v$ the WH model is ill-posed in any sense.

Recap

- GR in all Bondi-like gauges $\to$ weakly hyperbolic PDE system
  The root: pure gauge structure $g^{u\theta} = g^{u\phi} = 0$



- Ill-posed characteristic initial boundary value problem in the L^2 norm

- Accuracy of numerical results e.g. waveforms?

An open question

GR formulations with up to 3rd order metric derivatives (Newman-Penrose), can provide SH PDE system in Bondi-like gauges[1].

This CIBVP is well-posed in the L^2 norm.

Question: What does this mean for the CIBVP of the previous systems (up to 2nd order derivatives)?

[1] Rácz 2013; Cabet, Chruściel & Wafo 2014; Hilditch, Kroon & Peng 2019; Ripley 2020

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Thank you!

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Convergence tests

- Accuracy of numerical solution: $f - f_h = O(h^n) \rightarrow ||f - f_h|| = O(h^n)$
- Convergence factor: $Q = (h_c^n - h_m^n)/(h_m^n - h_f^n) = (f_c - f_m)/(f_m - f_f)$
- Solve the same PDE problem with increasing resolution

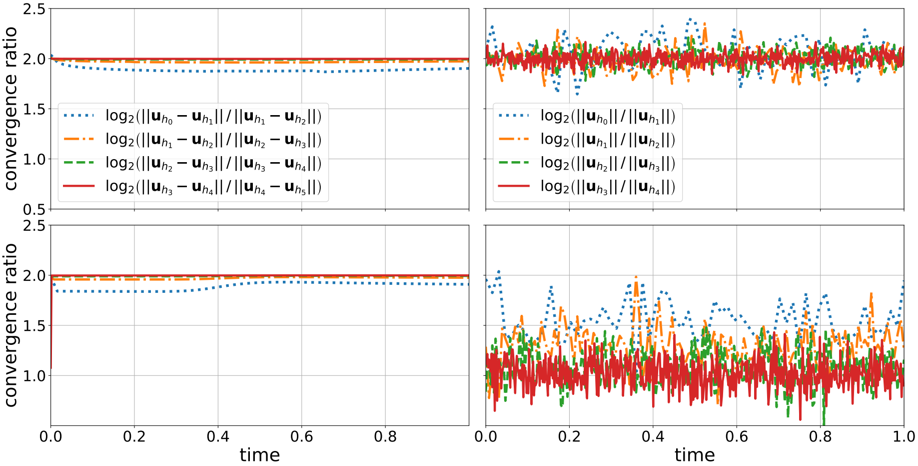
Smooth data (L^2 norm)

Noisy data of amplitude A
(L^2 & lopsided norm)

What is the behaviour of numerical error with increasing resolution?

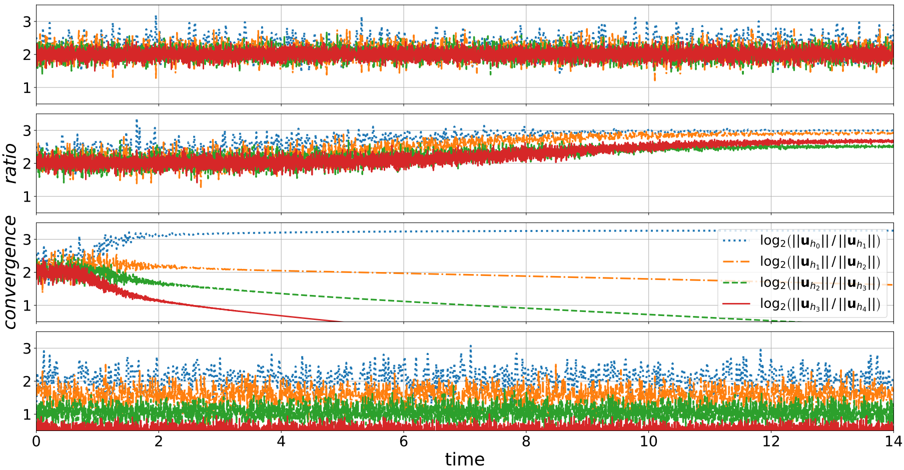
Calabrese, Hinder & Husa 2006

In the L^2 norm



smooth (left) & noisy (right) given data

In the lopsided norm



Homogeneous (1st) & inhomogeneous (2nd-4th) weakly hyperbolic models.

Algebraic determination of well-posedness

For the initial value problem (IVP) with constant coefficients:

$$\partial_t \mathbf{u} = \mathbf{B}^p \partial_p \mathbf{u} + \mathbf{S} \equiv \mathbf{B}^p \partial_p \mathbf{u} + \mathbf{B}\mathbf{u}\,,$$

after Fourier transforming in space ($\partial_p \to i\omega_p$):

$$\mathbf{P}(i\omega) = i\omega_p \mathbf{B}^p + \mathbf{B} \quad \longrightarrow \quad \mathbf{u}(\cdot\,, t) = e^{\mathbf{P}(i\omega)t}\hat{f}(\omega)\,.$$

If $|e^{\mathbf{P}(i\omega)t}| \leq Ke^{\alpha t}$, $K \geq 1$ & $\alpha \in \mathbb{R}$ for $t \geq 0$, the IVP is well posed in L^2.

$$||\mathbf{u}(\cdot\,, t)||_{L^2} = ||e^{\mathbf{P}(i\omega)t}\hat{f}(\omega)||_{L^2} \leq Ke^{\alpha t}||\hat{f}||_{L^2} = Ke^{\alpha t}||f||_{L^2}\,.$$

If $|e^{\mathbf{P}(i\omega)t}| \leq K_1 e^{\alpha t}\,(1 + |\omega|^q) \longrightarrow$ well-posed in a lopsided norm (weakly).

Eigenvalues of $\mathbf{P}(i\omega)$

Gustafsson, Kreiss & Oliger "*Time Dependent Problems and Difference Methods*"
Kreiss & Lorenz "*Initial-Boundary Value Problems and the Navier-Stokes Equations*"

Frame independence

Focus on the angular direction:

$$\partial_t \mathbf{u} + \mathbf{B}^{\hat{\theta}} \partial_{\hat{\theta}} \mathbf{u} \simeq 0 \quad \longrightarrow \quad \partial_t \mathbf{v} + \mathbf{J}^{\hat{\theta}} \partial_{\hat{\theta}} \mathbf{v} \simeq 0 \,,$$

where $\mathbf{J}^{\hat{\theta}} \equiv \mathbf{T}_{\hat{\theta}}^{-1} \, \mathbf{B}^{\hat{\theta}} \, \mathbf{T}_{\hat{\theta}}$ is the Jordan normal form and $\mathbf{v} \equiv \mathbf{T}_{\hat{\theta}}^{-1} \, \mathbf{u}$ the generalized characteristic variables. The non-trivial Jordan block yields

$$- \partial_t \left(2\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta \right) \simeq 0 \,,$$

$$\partial_t V - \partial_\theta \left(2\rho U + \frac{\rho^2}{2} U_r - \beta_\theta + \gamma_\theta \right) \simeq 0 \,.$$

The generalized eigenvalue problem:

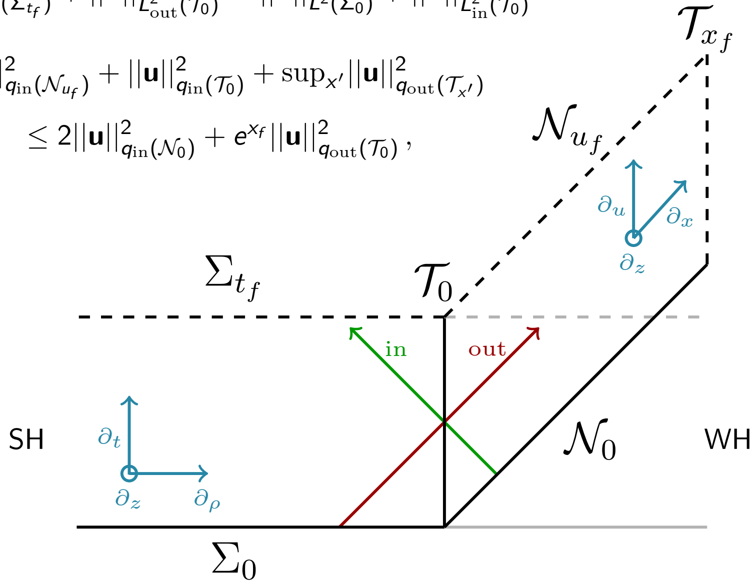
$$\mathbf{l}_\lambda \left(\mathbf{P}^s - \mathbf{1}\lambda \right)^M = 0 \,,$$

where $M = 1, 2, \cdots$.

Energy estimates

SH: $||\mathbf{u}||^2_{L^2(\Sigma_{t_f})} + ||\mathbf{u}||^2_{L^2_{\mathrm{out}}(\mathcal{T}_0)} = ||\mathbf{u}||^2_{L^2(\Sigma_0)} + ||\mathbf{u}||^2_{L^2_{\mathrm{in}}(\mathcal{T}_0)}$

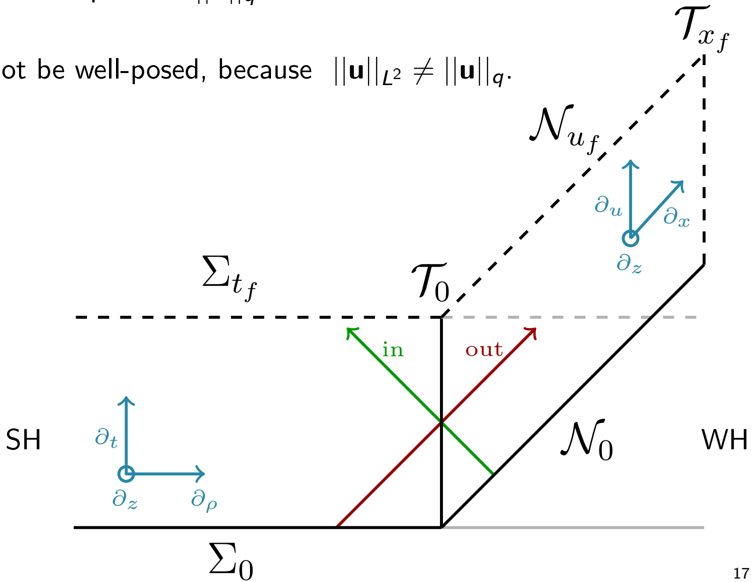
WH: $$2||\mathbf{u}||^2_{q_{\mathrm{in}}(\mathcal{N}_{u_f})} + ||\mathbf{u}||^2_{q_{\mathrm{in}}(\mathcal{T}_0)} + \sup_{x'} ||\mathbf{u}||^2_{q_{\mathrm{out}}(\mathcal{T}_{x'})} \leq 2||\mathbf{u}||^2_{q_{\mathrm{in}}(\mathcal{N}_0)} + e^{x_f}||\mathbf{u}||^2_{q_{\mathrm{out}}(\mathcal{T}_0)}\,,$$



Energy estimates, model CCE & CCM

CCE may be well-posed if $||\mathbf{u}||_q$ exists.

CCM cannot be well-posed, because $||\mathbf{u}||_{L^2} \neq ||\mathbf{u}||_q$.



- Accuracy of numerical solution: $f - f_h = O(h^n) \rightarrow ||f - f_h|| = O(h^n)$
- Convergence factor: $Q = (h_c^n - h_m^n)/(h_m^n - h_f^n) = (f_c - f_m)/(f_m - f_f)$
- Smooth data: $\mathcal{C}_{\mathrm{self}} = \log_2 \frac{||\mathbf{u}_{h_c} - \perp_{h_c}^{h_c/2} \mathbf{u}_{h_c/2}||_{h_c}}{||\perp_{h_c}^{h_c/2} \mathbf{u}_{h_c/2} - \perp_{h_c}^{h_c/4} \mathbf{u}_{h_c/4}||_{h_c}} = 2$
- Noisy data: $\mathcal{C}_{\mathrm{exact}} = \log_2 \frac{||\mathbf{u}_{h_c} - \mathbf{u}_{\mathrm{exact}}||_{h_c}}{||\perp_{h_c}^{h_c/2} \mathbf{u}_{h_c/2} - \mathbf{u}_{\mathrm{exact}}||_{h_c}}$

L^2 norm: $\log_2 \frac{O(A_{h_c})}{O(A_{h_c/2})}$

Lopsided norm: $\log_2 \frac{O(A_{h_c})}{2O(A_{h_c/2})}$