Gauge structure of GR in Bondi-like coordinates

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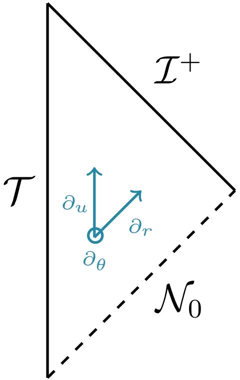
PRD.102.064035 TG, D. Hilditch, M. Zilhão
PRD.105.084055 TG, N. Bishop, D. Hilditch, D. Pollney, M. Zilhão

Bondi-like coordinates



- coordinates: u, r, θ, ϕ
- vector basis: $\partial_u\,, \partial_r\,, \partial_\theta\,, \partial_\phi$
- ∂_r is null & $\perp$ to ∂_θ and ∂_ϕ

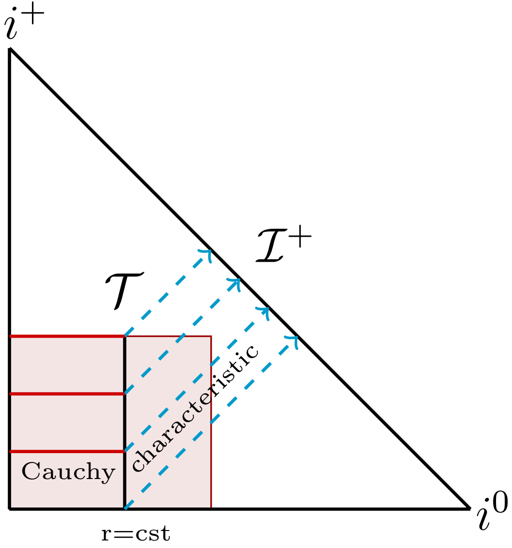
$$g^{\mu\nu} = \begin{pmatrix} 0 & g^{ur} & 0 & 0 \\ g^{ur} & g^{rr} & g^{r\theta} & g^{r\phi} \\ 0 & g^{r\theta} & g^{\theta\theta} & g^{\theta\phi} \\ 0 & g^{r\phi} & g^{\theta\phi} & g^{\phi\phi} \end{pmatrix}$$

Vacuum Einstein's equations:

Evolution system: $R_{rr} = R_{r\theta} = R_{r\phi} = R_{\theta\theta} = R_{\theta\phi} = R_{\phi\phi} = 0$

Bondi, van der Burg, Sachs 1962, Winicour 2013, Cao, He 2013

Highly accurate gravitational waveform modeling



Cauchy-Characteristic Extraction (CCE) & Matching (CCM)

see e.g. Winicour's 2012 Living Review and references therein

Why gauge structure?

It is related to hyperbolicity & well-posedness.

Hyperbolicity

$$\mathcal{A}^t(x^\mu)\,\partial_t\mathbf{u} + \mathcal{A}^p(x^\mu)\,\partial_p\mathbf{u} + \mathcal{S}(\mathbf{u},x^\mu) = 0\,,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_q)^T$, is the state vector of the system and

$$\mathcal{A}^\mu = \begin{pmatrix} a^\mu_{11} & \ldots & a^\mu_{1q} \\ \vdots & \ddots & \vdots \\ a^\mu_{q1} & \ldots & a^\mu_{qq} \end{pmatrix}$$

denotes the principal part matrices, with $\det(\mathcal{A}^t) \neq 0$. Construct the

$$\mathbf{P}^s = \left(\mathcal{A}^t\right)^{-1}\mathcal{A}^p\,s_p\,,$$

where s^i is an arbitrary unit spatial vector.

Well-posedness

The PDE problem has a unique solution that depends continuously on the given data in a suitable norm.

- Strongly hyperbolic (SH) $\rightarrow$ **well-posed** IVP in the L^2 norm
- Weakly hyperbolic (WH) $\rightarrow$ **ill-posed** IVP in the L^2 norm

A numerical solution **can converge** to the continuum **only** for well-posed PDE problems.

Gauge structure of GR

The ADM equations linearized about Minkowski:

$$\partial_t \delta\gamma_{ij} = -2\delta K_{ij} + \partial_{(i}\delta\beta_{j)}\,,$$
$$\partial_t \delta K_{ij} = -\partial_i\partial_j\delta\alpha - \tfrac{1}{2}\partial^k\partial_k\delta\gamma_{ij} - \tfrac{1}{2}\partial_i\partial_j\delta\gamma + \partial^k\partial_{(i}\delta\gamma_{j)k}\,.$$

First order reduction $\mathbf{u} = (\delta\gamma_{ij}, \delta\alpha, \delta\beta_i, \delta K_{ij}, \partial_p\delta\gamma_{ij}, \partial_p\delta\alpha, \partial_p\delta\beta_i)^T$:

$$\partial_t \mathbf{u} \simeq \mathbf{P}^s \partial_s \mathbf{u}\,, \quad \text{with} \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & 0 \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}.$$

The choice of α, β_i dictates the structure of $\mathbf{P}_G$.

Hilditch, Richter 2016; Khokhlov, Novikov 2001

Gauge structure of GR in Bondi-like coordinates

Non-diagonalizable $\mathbf{P}_G$ along the θ, ϕ directions[1].

$$g^{u\theta} = g^{u\phi} = 0$$

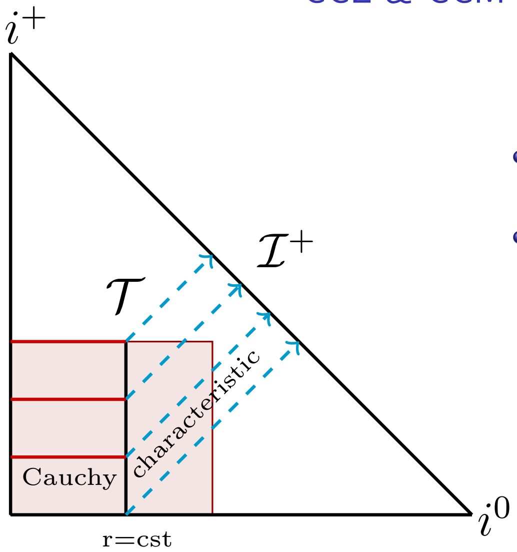
Mapping between Bondi-like and ADM equations.

GR formulation with up to 2nd order metric derivatives:

- In any Bondi-like gauge the PDE system is only WH.
- This CIBVP is ill-posed in the L^2 norm.
- This CIBVP **may** be weakly well-posed in a different norm[2,3].

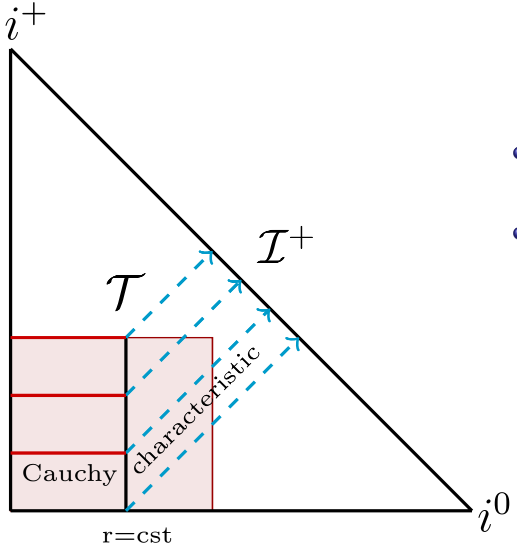
[1] TG, Bishop, Hilditch, Pollney, Zilhão 2022
[2] TG, Hilditch, Zilhão 2020
[3] Rácz 2013; Cabet, Chruściel, Wafo 2014; Hilditch, Kroon, Peng 2019; Ripley 2020

CCE & CCM



- Cauchy: SH PDE, well-posed in $||\mathbf{u}||_{L^2}$
- Characteristic: WH PDE, well-posed in $||\mathbf{u}||_q$

CCE & CCM



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Thank you!