Numerical Relativity: more formulations

Thanasis Giannakopoulos

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University of Nottingham
UK | CHINA | MALAYSIA

Lecture 1: A recipe for NR

1. Physical problem
2. **Formulation**
3. **PDE analysis**
4. Numerical methods
5. Implementation
6. Evaluate errors
7. Physical interpretation

Lecture 2: 3+1 formalism

1. 3+1 spacetime foliation
2. ADM system
3. BSSN formalism

Today

1. Generalized Harmonic Gauge (GHG) formulation
2. Characteristic formulations

GHG formulation

Used in the first successful simulation of full BBH merger (Pretorius 2005):

PRL 95, 121101 & CQG 22 425-452

Mathematical relativity:

Choquet-Bruhat 1962, The Cauchy problem

Numerics:

Szilágyi, Schmidt, Winicour 2002, PRD 65, 064015

Garfinkle 2002, PRD 65, 044029

Lindblom et al 2006, CQG 23 S447

BAMPS, SpEC & SpECTRE codes

The generalized harmonic evolution system

The vacuum EFE:

$$R_{ab} = -\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} + \boxed{\nabla_{(a}\Gamma_{b)}} + Q_{ab}(g, \partial g) = 0\,,$$

$$\Gamma_a = g_{ad} g^{bc} \Gamma^d{}_{bc} = \frac{1}{2} g^{bc} \left(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}\right)\,.$$

Generalized harmonic coordinate condition:

$$-\Gamma_a(g, \partial g) = g_{ab} \nabla^c \nabla_c x^b = H_a(x, g)\,.$$

The vacuum EFE in the generalized harmonic formulation:

$$-\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} - \boxed{\nabla_{(a} H_{b)}} + Q_{ab}(g, \partial g) = 0\,,$$

is manifestly symmetric hyperbolic.

Choosing GH coordinates

Specifying the GH coordinates means choosing $H_a(x, g)$:

- Standard harmonic choice: $H_a = 0$
- Generalized harmonic (algebraic): $H_a = f_a(x, g)$
  e.g. Lindblom et al 2006
- Promote H_a to independent variables (evolution eqs.): $\mathcal{L}_a H_a = 0$
  e.g. Pretorius 2005

How to choose the coordinates? The GH coordinate condition in 3+1 language:

$$\partial_t \alpha - \beta^k \partial_k \alpha = -\alpha \left(H_t - \beta^i H_i + \alpha K \right) ,$$

$$\partial_t \beta^i - \beta^k \partial_k \beta^i = \alpha \gamma^{ij} \left[\alpha \left(H_j + \gamma^{kl} \Gamma_{jkl} \right) - \partial_j \alpha \right] .$$

Constraints and reductions

$$0 = R_{ab} - \nabla_{(a}C_{b)} = -\frac{1}{2}g^{cd}\partial_c\partial_d g_{ab} - \nabla_{(a}H_{b)} + Q_{ab}(g, \partial g)$$

The GH coordinate constraint: $C_a \equiv H_a + \Gamma_a = 0$

- For ID that satisfy $C_a = \partial_t C_a = 0$ or $C_a = M_a = 0$, where $M_a = \{H, M_i\}$, the solution satisfies the GH constraint.
  - Lindblom et al. 2006
- Constraint violation: $0 = R_{ab} - \nabla_{(a}C_{b)} + \text{constraint damping terms}$
  - Pretorius used a suggestion by Gundlach et al. 2005

The reduction choice affects the numerical implementation:

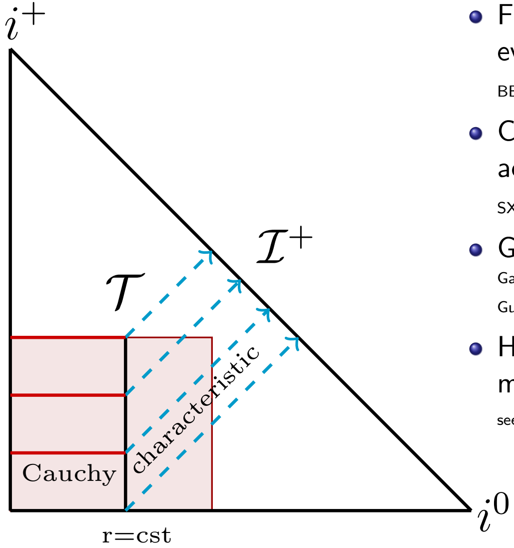
- Garfinkle 2002: 1st order in time, 2nd in space
- Pretorius 2005: 2nd order in time & space
- Lindblom et al. 2006: 1st order in time & space

Code	Open Source	Catalog	Formulation	Hydro	Beyond GR
AMSS-NCKU [43–46]	Yes	No	BSSN/Z4c	No	Yes
BAM [47–49]	No	[18]	BSSN/Z4c	Yes	No
BAMPS [50, 51]	No	No	GHG	Yes	No
COFFEE[52, 53]	Yes	No	GCFE	No	Yes
Dendro-GR [54–56]	Yes	No	BSSN/CCZ4	No	Yes
Einstein Toolkit [57, 58]	Yes	No	BSSN/Z4c	Yes	Yes
*Canuda [59–62]	Yes	No	BSSN	No	Yes
*IllinoisGRMHD [63]	Yes	No	BSSN	Yes	No
*LazEv [37, 64]	No	[65–68]	BSSN+CCZ4	No	No
*Lean [69, 70]	Partially	No	BSSN	No	Yes
*MAYA [71]	No	[71]	BSSN	No	Yes
*NRPy+ [72]	Yes	No	BSSN	Yes	No
*SphericalNR [73, 74]	No	No	spherical BSSN	Yes	No
*THC [75–77]	Yes	[18]	BSSN/Z4c	Yes	No
ExaHyPE [78]	Yes	No	CCZ4	Yes	No
FIL[79]	No	No	BSSN/Z4c/CCZ4	Yes	No
FUKA [80, 81]	Yes	No	XCTS	Yes	No
GR-Athena++ [82]	Yes	No	Z4c	Yes	No
GRChombo [83–85]	Yes	No	BSSN+CCZ4	No	Yes
HAD [86–88]	No	No	CCZ4	Yes	Yes
Illinois GRMHD [89, 90]	No	Yes	BSSN	Yes	No
MANGA/NRPy+ [91]	Partially	No	BSSN	Yes	No
MHDuet [92, 93]	No	No	CCZ4	Yes	Yes
SACRA-MPI [94]	No		BSSN+Z4c	Yes	No
SpEC [95, 96]	No	[96, 97]	GHG	Yes	Yes
SpECTRE [98, 99]	Yes	No	GHG	Yes	No
SPHINCS_BSSN [100]		No	BSSN	SPH	No

Table 1: List of numerical relativity codes. We indicate if a code is open-source, if it has been used to produce gravitational waveform catalogs, the formulation of Einstein's equation used (GHG: generalized harmonic, BSSN: Baumgarte-Shapiro-Shibata-Nakamura, CCZ4 / Z4c variants of the Z4 formulation, GCFE: generalised conformal field equations), if a code implements general relativistic hydrodynamics, and if it is capable to simulate compact binaries beyond general relativity. An asterisk indicates codes that are either (partially) based on the open-source Einstein Toolkit or are co-funded by its grant. Credit: Deidre Shoemaker; taken from Ref. [26].

Foucart, Laguna, Lovelace, Radice, Witek: Snowmass2021 Cosmic Frontier White Paper: Numerical relativity for next-generation gravitational-wave probes of fundamental physics

Characteristic formulations



- First stable numerical evolutions of a single BH

  BBHGC Alliance 1998
- CCE & CCM for highly accurate GW modeling

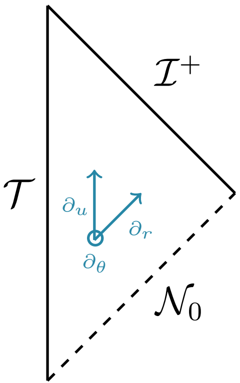
  SXS collaboration
- Gravitational collapse

  Garfinkle 1994, Crespo et al. 2019, Gundlach et al. 2022
- Holography & strongly coupled matter

  see e.g Chesler & Yaffe: arXiv:1309.1439,

Winicour's 2012 Living Review

Bondi-like coordinates



- coordinates: u, r, θ, ϕ
- ∂_r is null & $\perp$ to ∂_θ and ∂_ϕ

$$g^{\mu\nu} = \begin{pmatrix} 0 & g^{ur} & 0 & 0 \\ g^{ur} & g^{rr} & g^{r\theta} & g^{r\phi} \\ 0 & g^{r\theta} & g^{\theta\theta} & g^{\theta\phi} \\ 0 & g^{r\phi} & g^{\theta\phi} & g^{\phi\phi} \end{pmatrix}$$

Vacuum EFE:

Evolution system: $R_{rr} = R_{r\theta} = R_{r\phi} = R_{\theta\theta} = R_{\theta\phi} = R_{\phi\phi} = 0$

Constraints on $\mathcal{T}$: $R_{uu} = R_{u\theta} = R_{u\phi} = 0$ & trivial eq. $R_{ur} = 0$

Determinant condition: $g_{\theta\theta} g_{\phi\phi} - g_{\theta\phi}^2 = \hat{R}^4 \sin^2\theta$

Bondi-Sachs: $\hat{R} = r$, double null: $g^{rr} = 0$, affine null: $g^{rr} = \pm 1$

Bondi, van der Burg, Sachs 1962, Cao, He 2013

A characteristic PDE system

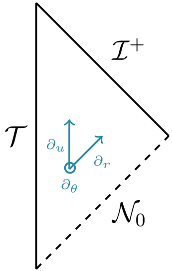
$$\mathcal{A}^t \partial_t \mathbf{u} + \mathcal{A}^p \partial_p \mathbf{u} + \mathcal{S} = 0\,,$$

with state vector $\mathbf{u} = (u_1, \ldots, u_q)^T$, coord. $x^\mu = (t, x^p)$ and "time" t

- $\mathrm{rank}\,(\mathcal{A}^t) = m < q$ and $\det\,(\mathcal{A}^t) = 0$
- $m - q$ (intrinsic) equations with no ∂_t and m with ∂_t
- Remember: for hyperbolicity we need the principal symbol $\mathbf{P}^s = (\mathcal{A}^t)^{-1}\,\mathcal{A}^p\, s_p$
- Use an auxiliary Cauchy-type system to study the hyperbolicity of a characteristic one

Example: Bondi-Sachs in axisymmetry

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du\, dr$$
$$+ 2Ur^2 e^{2\gamma}\, du\, d\theta - r^2 \left(e^{2\gamma}\, d\theta^2 + e^{-2\gamma} \sin^2\theta\, d\phi^2\right)$$



The Bondi-Sachs free evolution system:

$$\partial_r \beta = F_1(\partial_r \gamma)\,,$$
$$\partial_r^2 U = F_2(\gamma, \beta, \partial_i\gamma, \partial_i\beta, \partial_{ij}^2\gamma, \partial_{ij}^2\beta)\,,$$
$$\partial_r V = F_3(\gamma, \beta, \partial_i\gamma, \partial_i\beta, \partial_i U, \partial_{ij}^2\gamma, \partial_{ij}^2\beta, \partial_{ij}^2 U)\,,$$
$$\partial_{ur}^2 \gamma = F_4(\gamma, \beta, U, V, \partial_i\gamma, \partial_i\beta, \partial_i U, \partial_i V, \partial_{ij}^2\gamma, \partial_{ij}^2\beta, \partial_{ij}^2 U)$$

- Initial data: γ on $\mathcal{N}_0$, boundary data: β, U, V on $\mathcal{T}$
- Nested structure of intrinsic equations

Well-posedness of the Bondi-like CI(B)VP

- Existence and uniqueness of solutions: Frittelli & Lehner 1999, Gomez & Frittelli 2003
- Continuous dependence of the solution on the given data considering a subsystem: Frittelli 2005
- Weak hyperbolicity of the axisymmetric Bondi-Sachs system: TG, Hilditch, Zilhão 2020
- Weak hyperbolicity of GR in Bondi-like coordinates: TG, Bishop, Hilditch, Pollney, Zilhão 2021

  - $g^{u\theta} = g^{u\phi} = 0 \longrightarrow \mathbf{P}^{\theta}, \mathbf{P}^{\phi}$ are non-diagonalizable

The Bondi-like CI(B)VP is ill-posed in the L^2 norm.

Open question: Are there alternative norms?

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- Spacelike formulations (GHG)
- Characteristic formulations

Next

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- Toy models: implementation and convergence

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Thank you!