Numerical Relativity: 3+1 formalism

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Recap

The Einstein Field Equations (EFE) in geometric units ($G = c = 1$):

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}$$

Numerical Relativity (NR):

The use of numerical methods to find approximate solutions to the EFE, which converge to the continuum solution with more computational resources.

Recap: A recipe for NR

1. Physical problem
2. **Formulation**
3. **Analysis of partial differential equations (PDEs)**
4. Numerical methods
5. Implementation
6. Evaluate errors
7. Physical interpretation

Today

1. 3+1 spacetime foliation
2. ADM system
3. Hyperbolicity and well-posedness
4. Free evolution
5. BSSN formalism

3+1 spacetime foliation

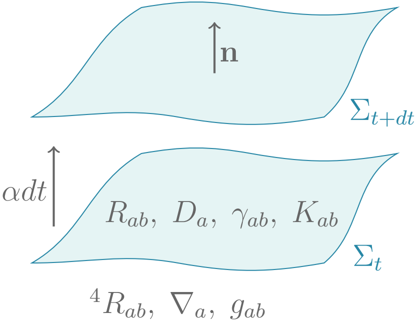
Goal: Solve the initial value problem (IVP) to get g_{ab}.

$$^4R_{ab} - \frac{1}{2} g_{ab} {}^4R + \Lambda g_{ab} = 8\pi T_{ab} \quad \longrightarrow \quad \partial_t \mathbf{u} = \mathbf{A}^p \partial_p \mathbf{u} + \mathbf{S}$$

- lapse α: $d\tau = \alpha dt$
- projection: $\gamma^a{}_b = \delta^a{}_b + n^a n_b$
- $K_{ab} \equiv -\gamma^c{}_a \nabla_c n_b = -\frac{1}{2}\mathcal{L}_{\mathbf{n}} \gamma_{ab}$

e.g. take an arbitrary 4-vector **V**:

$V^a = \delta^a{}_b V^b = \gamma^a{}_b V^b - n^a n_b V^b$



Alcubierre: Introduction to 3+1 Numerical Relativity
Gourgoulhon: 3+1 Formalism and Bases of Numerical Relativity arXiv:gr-qc/0703035v1

3+1 split the curvature

- Gauss equation: total projection onto Σ_t

$$\gamma^e{}_a \gamma^f{}_b \gamma^g{}_c \gamma^h{}_d \, {}^4R_{efgh} = R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc}$$

- Codazzi-Mainardi equation: project 1 with n and 3 with γ

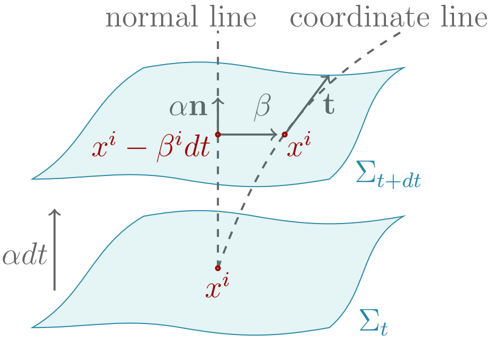
$$\gamma^e{}_a \gamma^f{}_b \gamma^g{}_c n^d \, {}^4R_{efgd} = D_b K_{ac} - D_a K_{bc}$$

- Ricci equation: project 2 with n and 2 with γ

$$\gamma^e{}_a n^b \gamma^g{}_c n^d \, {}^4R_{ebgd} = \mathcal{L}_{\mathbf{n}} K_{ac} + \frac{1}{\alpha} D_a D_c \alpha + K_{ad} K^d{}_c$$

3+1 adapted coordinates

- coordinates $x^\mu = (t, x^i)$
- $t^\mu = (1, \mathbf{0})^T$
- $\beta^\mu = (0, \beta^i)^T$
- $n^\mu = \alpha^{-1}(1, -\beta^i)^T$
- $g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$



3+1 split the EFE (with $\Lambda = 0$)

$$ {}^4R_{ab} - \frac{1}{2} g_{ab} {}^4R = 8\pi T_{ab} \,, \quad {}^4R_{ab} = 8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right) $$

Decompose T_{ab}: $\rho \equiv n^a n^b T_{ab} \,, \quad j_a \equiv -n^b \gamma^c{}_a T_{bc} \,, \quad S_{ab} \equiv \gamma^c{}_a \gamma^d{}_b T_{cd}$

Hamiltonian constraint (n projection): $R + K^2 - K_{ab}K^{ab} = 16\pi\rho$

Momentum constraints (n, γ projection): $D_b K^b{}_a - D_a K = 8\pi j_a$

Evolution equations (complete γ projection):

$$ \mathcal{L}_{\mathbf{n}} K_{ab} = -\alpha^{-1} D_a D_b \alpha + R_{ab} + K K_{ab} - 2 K_a{}^c K_{bc} - 8\pi \left[S_{ab} - \tfrac{1}{2} \gamma_{ab} (S - \rho) \right] $$

The ADM (or York) system

Use the $3+1$ adapted coordinates:

Hamiltonian constraint: $H \equiv R + K^2 - K_{ij}K^{ij} - 16\pi\rho = 0$

Momentum constraints: $M_j \equiv D_j K^j{}_i - D_i K - 8\pi j_i = 0$

K_{ij} evolution equations:

$$\partial_t K_{ij} = \mathcal{L}_\beta K_{ij} - \alpha^{-1} D_i D_j \alpha + R_{ij} + K K_{ij} - 2 K_i{}^l K_{jl}$$
$$-8\pi \left[S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho) \right] - \boxed{\frac{\alpha}{4} H}$$

γ_{ij} evolution equations: $\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$

Original goal: $\partial_t \mathbf{u} = \mathbf{A}^p \partial_p \mathbf{u} + \mathbf{S}$

- 10 EFE $\longrightarrow$ 6 evolution equations ($\partial_t K_{ij} = \dots$) + 4 constraints
- $\mathbf{u} = (\gamma_{ij}, K_{ij})^T$: hyperbolic PDE, with ∂_t, ∂_i^2, ∂_i, 12 evolved variables
- Evolution scheme: free vs constraint evolution
- Free evolution: solve $H, M_i|_{t=0}$ and evolve (γ_{ij}, K_{ij})
  - Bianchi identity $\rightarrow$ H, M_i satisfied for $t > 0$ if at $t = 0$ [Frittelli '97]
  - Constraint violation: i) purely numerical ii) constraint violating BC

Hyperbolicity

$$\boldsymbol{\mathcal{A}}^t(x^\mu)\,\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^p(x^\mu)\,\partial_p\mathbf{u} + \boldsymbol{\mathcal{S}}(\mathbf{u}, x^\mu) = 0\,,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_q)^T$, is the state vector of the system and

$$\boldsymbol{\mathcal{A}}^\mu = \begin{pmatrix} a^\mu_{11} & \dots & a^\mu_{1q} \\ \vdots & \ddots & \vdots \\ a^\mu_{q1} & \dots & a^\mu_{qq} \end{pmatrix}$$

denotes the principal part matrices, with $\det(\boldsymbol{\mathcal{A}}^t) \neq 0$. Construct the

$$\mathbf{P}^s = \left(\boldsymbol{\mathcal{A}}^t\right)^{-1} \boldsymbol{\mathcal{A}}^p\, s_p\,,$$

where s^i is an arbitrary unit spatial vector.

Well-posedness

The PDE problem has a unique solution that depends continuously on the given data in a suitable norm.

- Strongly hyperbolic (SH) $\rightarrow$ **well-posed** IVP in the L^2 norm
- Weakly hyperbolic (WH) $\rightarrow$ **ill-posed** IVP in the L^2 norm

A numerical solution **can converge** to the continuum **only** for well-posed PDE problems.

Free evolution

There is some freedom: i) addition of H, M_i, ii) choose α, β^i

$$H \equiv R + K^2 - K_{ij}K^{ij} = 0\,, \quad M_j \equiv D_j K^j{}_i - D_i K = 0$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$
$$\partial_t K_{ij} = \mathcal{L}_\beta K_{ij} - \alpha^{-1} D_i D_j \alpha + R_{ij} + K K_{ij} - 2K_i{}^l K_{jl} + \boxed{c_0 H + c_1 M_1}$$

- slicing (α) and coordinate evolution (β): algebraic, differential
  - hyperbolicity; physical problem; numerical cost
- geodesic slicing: $\alpha = 1$, $\beta^i = 0$ (WH ADM system)
  - Hahn and Lindquist 1964
- 1+log slicing: $(\partial_t - \mathcal{L}_\beta)\alpha = \alpha^2 K f(\alpha)$;
  - singularity avoidance properties; many BBH merger simulations

BSSN(OK)*

Widely used in many modern NR codes; good stability properties

Variables:

- conformal 3-metric: $\tilde{\gamma}_{ij} \equiv \psi^{-4}\gamma_{ij}$
  - nice for initial data and GWs
- traceless part of extr. curvature: $A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K$
- conformal rescale: $\tilde{A}_{ij} = \psi^{-4}A_{ij}$
- conformal connection functions $\tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk} = -\partial_j\tilde{\gamma}^{ij}$
  - important for stability (combined with momentum constraints)
- 17 evolved variables $(\psi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)^T$ vs 12 in ADM, $+\ \alpha, \beta^i$

*Baumgarte-Shapiro-Shibata-Nakamura-(Oohara and Kojima)

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- No: ID, BC, more ADM-like formulations & gauges

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Lecture 3

- Spacelike formulations (GHG)
- Characteristic formulations

Lecture 4

- Numerical methods (method of lines, finite differences)
- Toy models: implementation and convergence

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Thank you!