Hyperbolicity of GR in Bondi-like coordinates

Thanasis Giannakopoulos

University of Nottingham

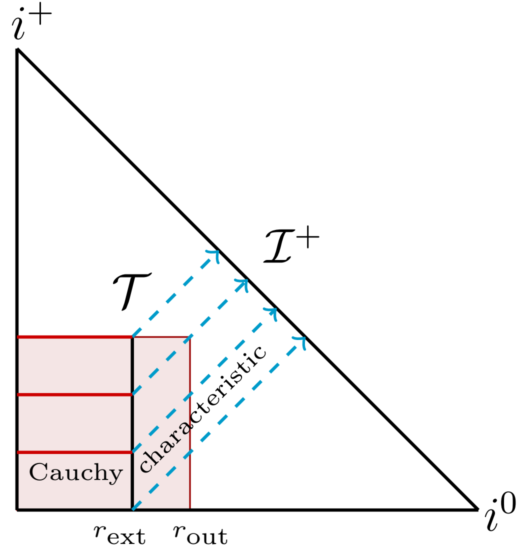
Gravity seminar, STAG Research Centre, Southampton,
22 February, 2024

Based on PRD.102.064035, PRD.105.084055, & PRD.108.104033
with N. Bishop, D. Hilditch, D. Pollney & M. Zilhão

Claim: **GR characteristic formulations with up to 2nd order metric derivatives in Bondi-like coordinates are weakly hyperbolic.**

- the characteristic problem in GR, hyperbolicity & well-posedness
- weak hyperbolicity of GR in Bondi-like coordinates, gauge structure, pure gauge subsystem
- well-posedness of Cauchy-characteristic extraction and matching

Highly accurate gravitational waveform modeling



Cauchy-Characteristic Extraction (CCE) and Matching (CCM)

see e.g. Winicour's 2012 Living Review and references therein

Hyperbolicity

$$\boldsymbol{\mathcal{A}}^t(x^\mu)\,\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^p(x^\mu)\,\partial_p\mathbf{u} + \boldsymbol{\mathcal{S}}(\mathbf{u}, x^\mu) = 0\,, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_q)^T$ is the state vector of the system and $\boldsymbol{\mathcal{A}}^\mu$ denotes the principal part matrices, with $\det(\boldsymbol{\mathcal{A}}^t) \neq 0$. Construct the principal symbol

$$\mathbf{P}^s = \left(\boldsymbol{\mathcal{A}}^t\right)^{-1}\boldsymbol{\mathcal{A}}^p\, s_p\,,$$

where s^i is an arbitrary unit spatial vector.

<u>Degree of hyperbolicity:</u>

- $\mathbf{P}^s$ has real eigenvalues for all s^i $\rightarrow$ (1) is weakly hyperbolic (WH)
- $\mathbf{P}^s$ is also diagonalizable for all s^i $\rightarrow$ (1) is strongly hyperbolic (SH)
- all $\boldsymbol{\mathcal{A}}^\mu$ are symmetric $\rightarrow$ (1) is symmetric hyperbolic (SYMH)

Gustafsson, Kreiss & Oliger "*Time Dependent Problems and Difference Methods*"
Kreiss & Lorenz "*Initial-Boundary Value Problems and the Navier-Stokes Equations*"

Well-posedness

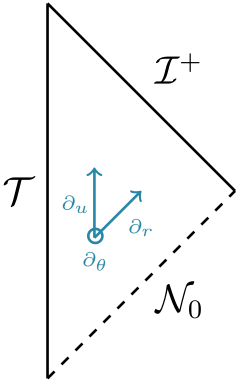
The PDE problem has a unique solution **u** that depends continuously on the given data f in a suitable norm $||\cdot||$:

$||\mathbf{u}|| \leq Ke^{\alpha t}||f||$, for real constants $K > 1$, α, and t.

- Strongly hyperbolic $\rightarrow$ **well-posed** IVP in the L^2 norm
- Weakly hyperbolic $\rightarrow$ **ill-posed** IVP in the L^2 norm
  possibly **weakly well-posed** in a different norm

A numerical solution **can converge** to the continuum **only** for well-posed PDE problems.

Bondi-like coordinates



- coordinates: u, r, θ, ϕ
- vector basis: $\partial_u\,, \partial_r\,, \partial_\theta\,, \partial_\phi$
- ∂_r is null & $\perp$ to ∂_θ and ∂_ϕ

$$g_{\mu\nu} = \begin{pmatrix} g_{uu} & g_{ur} & g_{u\theta} & g_{u\phi} \\ g_{ur} & 0 & 0 & 0 \\ g_{u\theta} & 0 & g_{\theta\theta} & g_{\theta\phi} \\ g_{u\phi} & 0 & g_{\theta\phi} & g_{\phi\phi} \end{pmatrix}$$

Vacuum Einstein's equations:

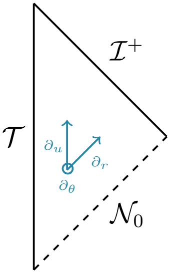
Evolution system: $R_{rr} = R_{r\theta} = R_{r\phi} = R_{\theta\theta} = R_{\theta\phi} = R_{\phi\phi} = 0$

Constraints on $\mathcal{T}$: $R_{uu} = R_{u\theta} = R_{u\phi} = 0$ & trivial eq.: $R_{ur} = 0$

Bondi, van der Burg & Sachs 1962; Winicour 2013; Cao & He 2013

Hyperbolicity of GR in the Bondi-Sachs gauge

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du\, dr$$
$$+ 2Ur^2 e^{2\gamma} du\, d\theta - r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma}\sin^2\theta\, d\phi^2\right) .$$



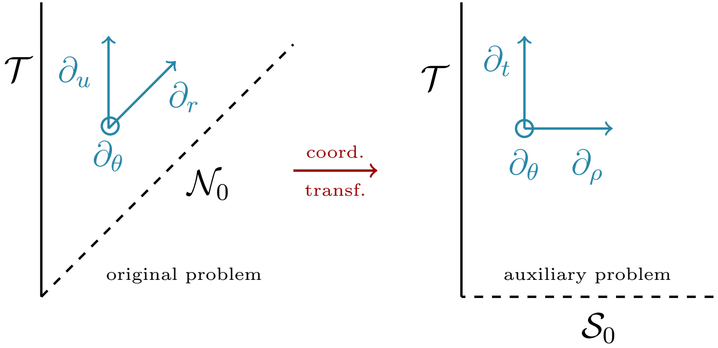
The PDE system:

$$\partial_r \beta = F_1(\partial_r \gamma)\,,$$
$$\partial_r^2 U = F_2(\gamma, \beta, \partial_i\gamma, \partial_i\beta, \partial_{ij}^2\gamma, \partial_{ij}^2\beta)\,,$$
$$\partial_r V = F_3(\gamma, \beta, \partial_i\gamma, \partial_i\beta, \partial_i U, \partial_{ij}^2\gamma, \partial_{ij}^2\beta, \partial_{ij}^2 U)\,,$$
$$\partial_{ur}^2 \gamma = F_4(\gamma, \beta, U, V, \partial_i\gamma, \partial_i\beta, \partial_i U, \partial_i V, \partial_{ij}^2\gamma, \partial_{ij}^2\beta, \partial_{ij}^2 U)$$

Linearize and first order reduction $\mathbf{u} = (\beta\,, \gamma\,, U\,, V\,, \gamma_r\,, U_r\,, \beta_\theta\,, \gamma_\theta)^T$:

$$\boldsymbol{\mathcal{A}}^u \partial_u \mathbf{u} + \boldsymbol{\mathcal{A}}^r \partial_r \mathbf{u} + \boldsymbol{\mathcal{A}}^\theta \partial_\theta \mathbf{u} + \boldsymbol{\mathcal{S}} = 0.$$

$$\det(\boldsymbol{\mathcal{A}}^u) = 0\,, \qquad \begin{aligned} u &= t - \rho\,, \\ r &= \rho\,, \end{aligned} \qquad \det(\boldsymbol{\mathcal{A}}^t) \neq 0\,.$$



$$\boldsymbol{\mathcal{A}}^t\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^\rho\partial_\rho\mathbf{u} + \boldsymbol{\mathcal{A}}^\theta\partial_\theta\mathbf{u} + \boldsymbol{\mathcal{S}} = 0, \text{ where } \boldsymbol{\mathcal{A}}^t = \boldsymbol{\mathcal{A}}^u + \boldsymbol{\mathcal{A}}^r \text{ and } \boldsymbol{\mathcal{A}}^\rho = \boldsymbol{\mathcal{A}}^r\,.$$

$$\mathbf{P}^\theta = \frac{1}{\rho}\left(\boldsymbol{\mathcal{A}}^t\right)^{-1}\boldsymbol{\mathcal{A}}^\theta \text{ is not diagonalizable.}$$

The Bondi-Sachs system is weakly hyperbolic.

Rendall 1990; Frittelli 2005 & 2006; TG, Hilditch & Zilhão 2020

Gauge structure of GR

The ADM equations linearized about Minkowski:

$$\partial_t \delta\gamma_{ij} = -2\delta K_{ij} + \partial_{(i}\delta\beta_{j)}\,,$$
$$\partial_t \delta K_{ij} = -\partial_i\partial_j\delta\alpha - \tfrac{1}{2}\partial^k\partial_k\delta\gamma_{ij} - \tfrac{1}{2}\partial_i\partial_j\delta\gamma + \partial^k\partial_{(i}\delta\gamma_{j)k}\,.$$

First order reduction $\mathbf{u} = (\delta\gamma_{ij}, \delta\alpha, \delta\beta_i, \delta K_{ij}, \partial_p\delta\gamma_{ij}, \partial_p\delta\alpha, \partial_p\delta\beta_i)^T$:

$$\partial_t \mathbf{u} \simeq \mathbf{P}^s\partial_s\mathbf{u}\,, \quad \text{with} \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & 0 \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}.$$

Hilditch & Richter 2016

Pure gauge subsystem

Assume an arbitrary solution $g_{\mu\nu}$ of $R_{\mu\nu} = 0$.

- Infinitesimal coordinate transformation: $x^\mu \to x^\mu + \xi^\mu$
- Perturbation to the solution: $\delta g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}$
- $3+1$ split: $\Theta \equiv n_\mu \xi^\mu$, $\psi^i \equiv -\gamma^i{}_\mu \xi^\mu$

Pure gauge subsystem for flat background:

$$\partial_t \Theta = \delta\alpha \,,$$
$$\partial_t \psi_i = \delta\beta_i + \partial_i \Theta \,.$$

Given α, β_i, the pure gauge subsystem is closed.

Khokhlov & Novikov 2001

Pure gauge subsystem inheritance

Linearized ADM system:

$$\partial_t \mathbf{u} \simeq \mathbf{P}^s \partial_s \mathbf{u}\,, \quad \mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & \mathbf{P}_{GC} \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}.$$

Assume an algebraic choice for α, β_i. Pure gauge subsystem:

$$\partial_t \mathbf{v}_{gauge} \simeq \mathbf{P}^s_{gauge} \partial_s \mathbf{v}_{gauge}\,, \quad v_{gauge} = (\Theta, \psi_i)^T\,.$$

The inheritance: $\mathbf{P}_G = \mathbf{P}^s_{gauge}$

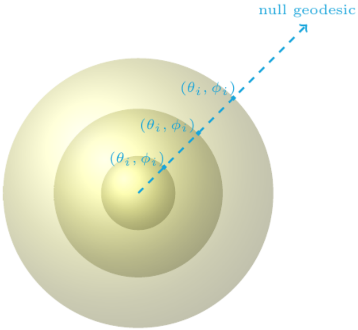
The result holds also for generic backgrounds & differential gauge choices.

Hilditch & Richter 2016

Gauge structure of Bondi-like coordinates

Mapping between Bondi-like and ADM equations
$\rightarrow$ Non-diagonalizable $\mathbf{P}_G$ along the θ, ϕ directions

$$g^{u\theta} = g^{u\phi} = 0$$

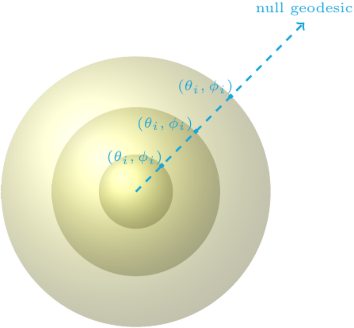


TG, Bishop, Hilditch, Pollney & Zilhão 2022

Gauge structure of Bondi-like coordinates

Mapping between Bondi-like and ADM equations
$\rightarrow$ Non-diagonalizable $\mathbf{P}_G$ along the θ, ϕ directions

$$g^{u\theta} = g^{u\phi} = 0$$



GR characteristic formulations with up to 2nd order metric derivatives in Bondi-like coordinates are weakly hyperbolic.
$\rightarrow$ Ill-posed CIBVP in the L^2 norm

TG, Bishop, Hilditch, Pollney & Zilhão 2022

Open questions

GR formulations with up to 3rd order metric derivatives (Newman-Penrose), can provide SYMH PDE system in Bondi-like gauges[1].

This CIBVP is well-posed in the L^2 norm.

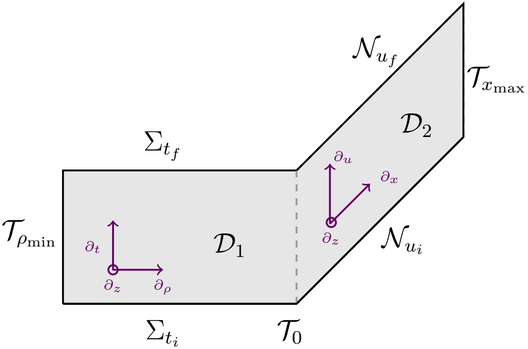
- Is there a relation between the 3rd and 2nd order systems?
  Is the CIBVP of the 2nd order setup well-posed in any norm?

- How does the WH of 2nd order setups affects the accuracy of CCE and CCM?

  Explore CCE and CCM with toy models[2].

[1] Rácz 2013; Cabet, Chruściel & Wafo 2014; Hilditch, Kroon & Peng 2019; Ripley 2020
[2] TG, Bishop, Hilditch, Pollney & Zilhão 2023

The toy models

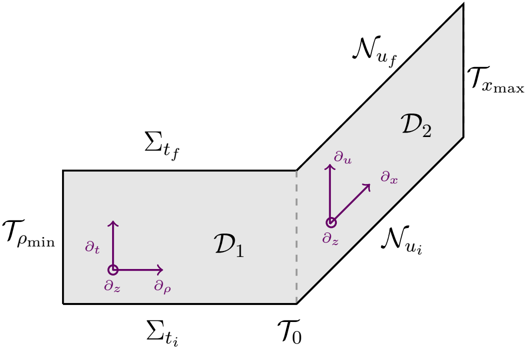


$$\partial_t \phi_1 = -\partial_\rho \phi_1 + \boxed{\partial_z \psi_{v1}} + \psi_1$$
$$\partial_t \psi_{v1} = -\partial_\rho \psi_{v1} + \partial_z \phi_1$$
$$\partial_t \psi_1 = \partial_\rho \psi_1 + \partial_z \psi_1$$

$$\partial_x \phi_2 = \boxed{\partial_z \psi_{v2}}$$
$$\partial_x \psi_{v2} = \partial_z \phi_2$$
$$\partial_u \psi_2 = \frac{1}{2}\partial_x \psi_2 + \partial_z \psi_2 + \psi_{v2}$$

SYMH when $\partial_z \psi_v$ is included, only WH otherwise

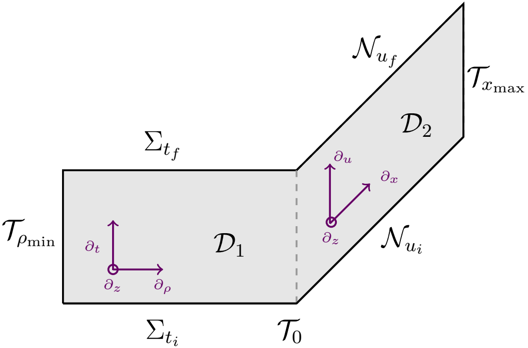
Energy estimates: SYMH



IBVP: $||\mathbf{u}_1||_{L^2} \equiv \int_{\Sigma_{t_f}} (\phi_1^2 + \psi_{v1}^2 + \psi_1^2) + \int_{\mathcal{T}_0} (\phi_1^2 + \psi_{v1}^2) + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2$

CIBVP: $||\mathbf{u}_2||_{L^2} \equiv \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_0} \psi_2^2 + \max_{x'} \int_{\mathcal{T}_{x'}} (\phi_2^2 + \psi_{v2}^2)$

Energy estimates: WH



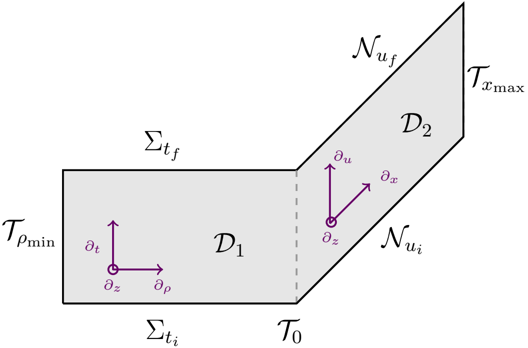
IBVP: $||\mathbf{u}_1||_q \equiv$

$$\int_{\Sigma_{t_f}} \left[\phi_1^2 + \psi_{v1}^2 + \psi_1^2 + \boxed{(\partial_z\phi_1)^2}\right] + \int_{\mathcal{T}_0} \left[\phi_1^2 + \psi_{v1}^2 + \boxed{(\partial_z\phi_1)^2}\right] + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2$$

CIBVP: $||\mathbf{u}_2||_q \equiv \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_0} \psi_2^2 + \max_{x'} \int_{\mathcal{T}_{x'}} \left[\phi_2^2 + \psi_{v2}^2 + \boxed{(\partial_z\phi_2)^2}\right]$

CCE well-posedness is examined individually for the IBVP and CIBVP.

Energy estimates: CCM



<u>SYMH-SYMH:</u>

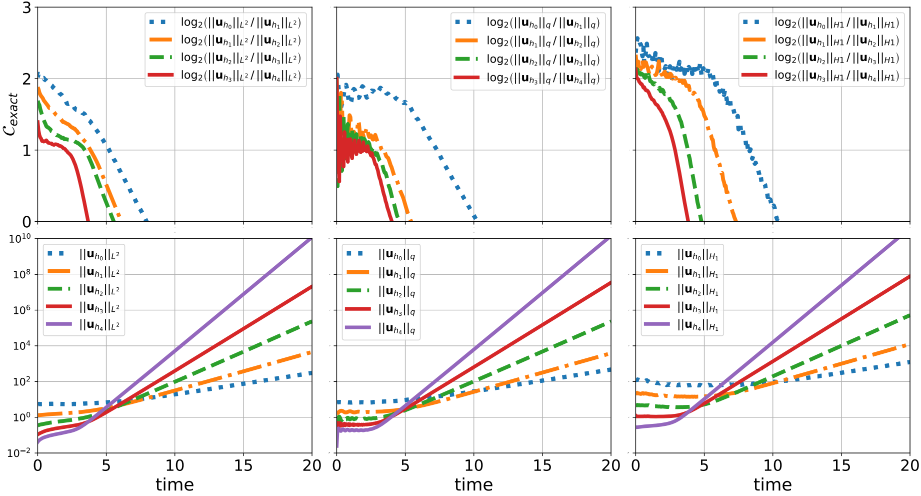
$$||\mathbf{u}||_{L^2} \equiv \int_{\Sigma_{t_f}} (\phi_1^2 + \psi_{v1}^2 + \psi_1^2) + \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2 + \max_{x'} \int_{\mathcal{T}_{x'}} (\phi_2^2 + \psi_{v2}^2)$$

<u>WH-WH:</u>

$$||\mathbf{u}||_q \equiv \int_{\Sigma_{t_f}} \left[\phi_1^2 + \psi_{v1}^2 + \psi_1^2 + \boxed{(\partial_z \phi_1)^2}\right] + \int_{\mathcal{N}_{u_f}} \psi_2^2 + \int_{\mathcal{T}_{\rho_{\min}}} \psi_1^2 + \max_{x'} \int_{\mathcal{T}_{x'}} \left[\phi_2^2 + \psi_{v2}^2 + \boxed{(\partial_z \phi_2)^2}\right]$$

Robust stability tests

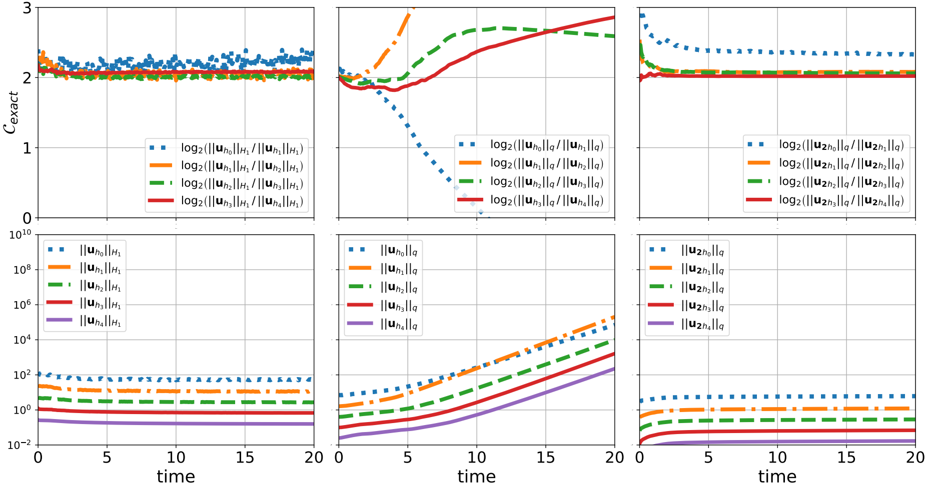
Test passed: $\mathcal{C}_{\text{exact}} \simeq 2$



CCM between the SYMH IBVP and the WH CIBVP in different norms

Robust stability tests

Test passed: $\mathcal{C}_{\text{exact}} \simeq 2$



CCM between the SYMH-SYMH (left), WH-WH (middle) and the WH CIBVP (right) for CCE between SYMH-WH

Conclusions

- 2nd order Bondi-like systems are weakly hyperbolic:
  -Ill-posed CIBVP in the L^2 norm
  -Weakly well-posed CIBVP in a different norm?
- If the CIBVP is weakly well-posed, CCE can also be well-posed
- CCM as currently performed (SYMH-WH) is ill-posed
- Problem with error estimates for accurate waveforms with CCM
- A strongly or symmetric hyperbolic characteristic formulation is needed (with up to 2nd order metric derivatives)

Conclusions

- 2nd order Bondi-like systems are weakly hyperbolic:
  -Ill-posed CIBVP in the L^2 norm
  -Weakly well-posed CIBVP in a different norm?
- If the CIBVP is weakly well-posed, CCE can also be well-posed
- CCM as currently performed (SYMH-WH) is ill-posed
- Problem with error estimates for accurate waveforms with CCM
- A strongly or symmetric hyperbolic characteristic formulation is needed (with up to 2nd order metric derivatives)

Thank you!